# **Understanding Geometric Pattern and its Geometry Part 2 – Decagonal Diversity**

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#### Abstract

In the first part of this paper<sup>1</sup>, we discussed selected geometric concepts related to the class of patterns that were referred to as gereh<sup>2</sup>. The creation of these patterns followed the rules of gereh (axioms). Constructions of such patterns comprised three steps: construction of the contour for the pattern template, construction of tessellation inside the contour, and designing a pattern for each tessellation tile.

In this part, we will discuss some concepts of decagonal gerehs, i.e., gerehs with tessellation tiles derived in some way from the geometry of a regular decagon. We will discuss different types of decagonal patterns, as well as tessellations used to create decagonal designs.

# Introduction

In this paper, we will discuss several types of decagonal patterns with Persian origins. For each of them, we will use a different term to describe the given type. However, we will not try to make any formal classification of decagonal patterns. The terms used in this paper should be understood as labels that we attach to a particular group of designs to distinguish them from another group. It works in the same way as when we compare objects in real life – we often say that a given thing looks like another given thing. A good example is a sentence "my car looks like the car of my neighbor; it is blue with scratches on one side."

In the past, there were some attempts to make a formal classification of geometric patterns with given local symmetries. For example, Anthony Lee tried to establish a classification of decagonal patterns (Lee, [2]). In this paper, we will come back to his classification, or let us say typology, and compare it with our observations.

As we will see later, there is no easy way to make a precise classification of geometric patterns inside the group of decagonal gerehs. Persian and later Ottoman artists created designs mixing different features. Thus while visiting mosques in Turkey, we may see designs combining Persian and Ottoman elements. There are also numerous gerehs with some non-typical geometric solutions. Two collections of such designs can be seen in Rustem Pasha Mosque and Sultan Selim Mosque both in Istanbul. Another splendid gereh on the main gate to the external park in Topkapi palace is often considered as an incorrect. In several places in the Middle East, we can see designs with some unusual features. Several of them will be discussed in another part of this paper.

All examples in this paper will be labeled as 'projects,' and each of them can be treated as a separate project in constructive geometry. Each of these projects can be done using compasses and a ruler or computer software. All drawings in this paper were done with Geometer's Sketchpad. All projects presented here were taken from my book (Majewski, [4]). Some of them were revised. The numbers of projects are the same as their numbers in the book. Thus we may have some gaps in project numbering.

<sup>&</sup>lt;sup>1</sup> Understanding Geometric Pattern and its Geometry (part 1), eJMT, vol. 14, Nr 2, pages 87-106.

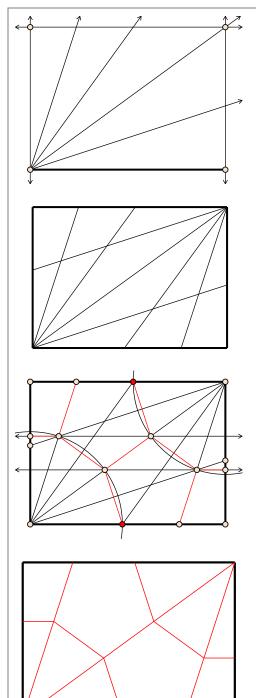
<sup>&</sup>lt;sup>2</sup> Gerehs, or gereh patterns, this is only one group of geometric patterns. There are many other patterns that can be created using different approach, e.g. patterns on square grids, triangular or other types of grids.

# A tessellation with two pentagons, trapeziums and decagons

There are countless tessellations for decagonal gerehs. The one discussed in this chapter is undoubtedly the most frequently used. Here we show its detailed construction.

# Project 2.1 - First decagonal contour and first tessellation

We will start by creating a typical contour for decagonal patterns. We will construct a tessellation based on this contour, and finally, we will create a few designs that can be built using this particular contour and this particular tessellation.



**Fig. 2.1A.** Start by drawing a horizontal segment. This will be the bottom line of the contour. At both ends of this segment, construct two lines perpendicular to it.

Divide one of the right angles into 5 equal parts.

Mark the point where the second section line from the bottom intersects with the right vertical line. Draw a line

bottom intersects with the right vertical line. Draw a line passing through this point and parallel to the bottom segment. The rectangle bounded by these lines is our first and the most popular contour for many decagonal patterns.

**FIG. 2.1B.** Draw edges of the contour, and divide the topright angle into 5 equal parts. This way, we will get the network shown here. Note, the whole network is based on the division of the right angle into 5 equal parts. These are the white lines on the photographs of scrolls of medieval architects.

**FIG. 2.1c.** Use the two red points on the edges of the contour to draw two circles with centers in opposite corners of the contour and passing through the red points. Then make knots where the 'strings' and circles intersect. All knots are marked here as small circles. Finally, connect these knots with red segments. This is our complete tessellation. Note how the short red segments near the right and left edge were created. Now we can remove all construction lines and leave only the most necessary elements, i.e., the network of red lines and the contour.

**FIG. 2.1D.** The final view of the tessellation created in this example. We have here two regular pentagons. We have here also two-quarters of two decagons located in the bottom-left and top-right corner. We divided them into long triangles. Finally, we have here two-quarters of trapeziums situated in the top-left and bottom-right corners.

Note also if we denote lengths of the sides of decagons and pentagons as S, then the long sides of triangles will have length L, where L/S= 1.61803= $\Phi$ . Where  $\Phi$  is the golden proportion.

# Styles of decagonal gerehs

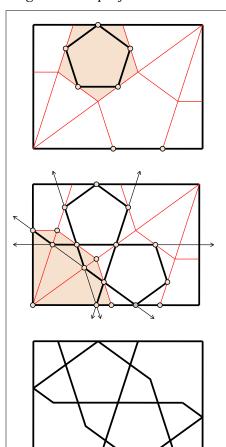
As it was mentioned before, the tessellation from this first project is the most popular tessellation for decagonal patterns. Now everything depends on how the first line of the future pattern will be placed. Here is one such design.

Project 2.2 - The Nodir Devon Madrasah style



**FIG. 2.2** A ceramic mosaic from the inner portal of the Nodir Devon Madrasah from Bukhara.

The goal of this project will be to reconstruct the pattern shown in the photograph.



**FIG. 2.2A.** Take one of the pentagons in our tessellation and mark the center of each of its sides. Now, draw segments connecting the centers of its adjoining sides. This way, we will get a nice smaller pentagon. We can do the same for the other red pentagon. This is your task.

**FIG. 2.2B.** After filling the two red pentagons with the pattern, we extend edges of this pattern onto the left quarter of the decagon. By following the thin ropes, we make a few more knots, and this way, we get the design inside the decagon. We do precisely the same for the other quarter of decagon and halves of trapeziums. The template for the Nodir Devon Madrasah pattern will be ready.

**Fig. 2.2c.** Finished template for the Nodir Devon Madrasah pattern. Although this is the simplest pattern that could be designed on the tessellation from project 1, we will see later that this template can be used to create several more complex patterns.

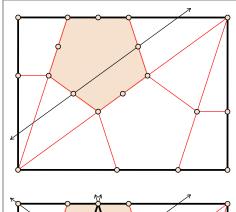
The style of the pattern shown in the photo (fig. 2.2) we will call the Nodir Devon style. There is no particular reason to give it this or any other name. We just need a name that can be used for future reference. We may use a more or less descriptive name for this style. Tony Lee, in his manuscript (Lee, [2]), refers to it as Type I. However, some of his later types are just variations of this one. We can refer to this particular style by the measure of angles used in it. But this can be a bit misleading – which angles should we use? The same pattern can be seen in many other places, e.g., in the Karatay Madrasah in Konya, the Tilia Kari Madrasah, and Bibi Khanum Madrasah in Samarkand. Thus we could use another name to identify this style. It does not matter what name we use as long as we know what kind of pattern we mean.

#### Project 2.3 - The Kukeldash Madrasah style

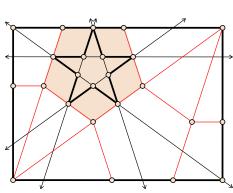


Fig. 2.3. The Kukeldash Madrasah design The photo shows the bottom part of the doors to the Kukeldash Madrasah in Bukhara in Uzbekistan. There are two designs on these doors. The other one, more complicated, we will discuss later. The Kukeldash Madrasah in Bukhara was built in (1568-1569), and the doors to it are the original doors made around 1569-70. Thus, this is probably one of the oldest wooden artifacts with such a design. This is also one of the oldest known examples of kundekari art.

We start with the same tessellation. This time we will place our first line in a slightly different way. We will see how the pattern changes depending on how the first line in the design is selected.

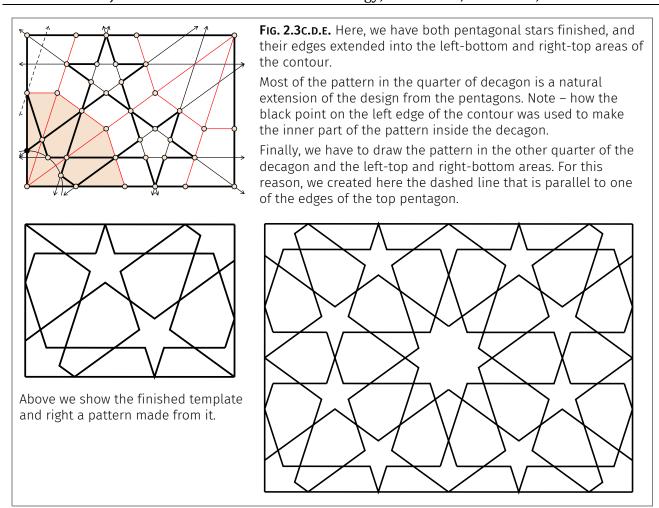


**FIG. 2.3A.** We start construction of the pattern inside a pentagon by drawing the first line through the two points shown in the figure. Of course, now, due to the symmetry rule G8, we have also to draw four similar lines passing through the other pairs of midpoints on sides of the pentagon.



**FIG. 2.3B.** After drawing the other construction lines, we get an excellent star with sharp ends. Its center can be filled with additional segments forming there a small pentagon. This way, we will get a slight variation of this gereh.

Now let us draw identical lines in the other pentagon, and we will get almost everything that we need to construct the complete template.

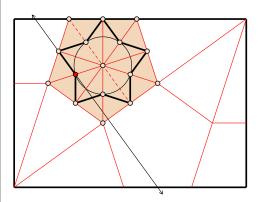


In the title for this project, I used the term 'Kukeldash Madrasah style.' Again there is no particular reason for selecting this or any other name. We need it only for reference purposes. The pattern shown above is one of the most popular designs in Uzbekistan, Iran, and Turkey. We can find it in various versions all over the Middle East. Its specific feature are the sharp angles in pentagonal stars. Tony Lee refers to this style as type II.

# Project 2.4 - The Persian style

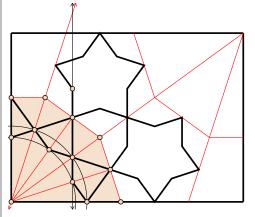
Fig 2.4. This particular photo was made in Isfahan Jame Mosque. It shows a fragment of a large ceramic mosaic on the wall. Patterns like this one are very popular all over Central Asia. In Uzbekistan, we may see it on the wall of one of madrasah in Regestan square in Samarkand and many other places. A very typical feature of this design are the pentagonal stars with parallel, or almost parallel, edges. This, of course, will influence all other shapes used in this design.





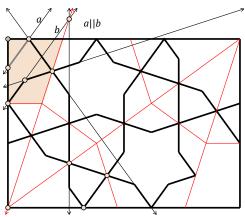
**FIG. 2.4A.** We start again with the same tessellation and midpoints on the sides of one of the pentagons. We draw all symmetry lines of the pentagon, and then from one of the edges of it, we construct a line passing through the center of the side and parallel to the dashed symmetry line. This is our first line.

We use the intersection point of the first line with one of the symmetry lines to draw the circle shown here. This way, we got a framework to draw a star pattern inside the pentagon. We should do the same with the other pentagon.



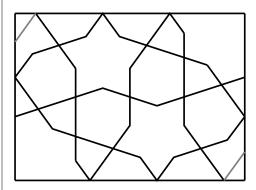
**FIG. 2.4B.** The pentagonal stars are ready. Now, we extend one of the edges of such a star. Just one is enough. We get a vertical line crossing some of the symmetry lines of the decagon in the left-bottom corner. Now we use the points of intersection of the vertical line with decagon symmetry lines to create the two circles. This is our framework for drawing the pattern inside the decagon.

We should repeat the same in the right-top corner.

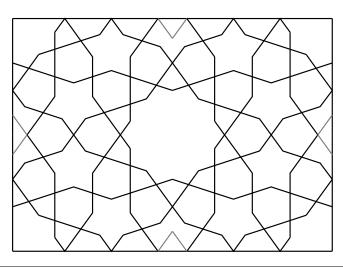


**FIG. 2.4c.D.E.** Here we show how one could proceed with the finishing touches in the half of the trapezium. Again we extend some lines from the pattern in the pentagon. Lines *a* and *b* in the left-top corner are parallel.

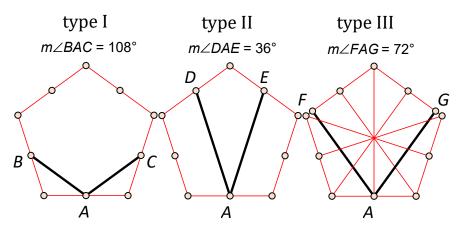
Note – this pattern may or not have the thick gray segment connecting two points in the top-left corner. In the design from Samarkand, we do not have it, but in many other places, it is included.



Above we show the finished template and to the right a pattern made from it. Here we show the version with the optional gray segments.



At the beginning of this project, I used the term 'Persian style.' Tony Lee refers to it as Type III. Let us look for a while at specific features of these three styles or types.

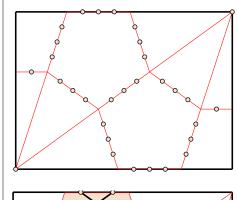


# Types of decagonal patterns according to Tony Lee

Type I has the widest angle, Type II has the narrower angle, and finally, Type III has a medium angle. The first type we called the Nodir Devon Madrasah style, the second one as the Kukeldash Madrasah style and the third one as the Persian style.

# Project 2.5 - The Tan Sahid Mosque style

This is not the most popular kind of pattern. However, we see it in a few places in Central Asia and Egypt, e.g., the Tan Sahid Mosque in Iran, Madrasa Amir Sungur Sa'di in Cairo, and also in Turkey in the Sunqur Bek mosque. A very beautiful pattern using this style can be seen in the Rustem Pasa Mosque in Istanbul.



**FIG. 2.5A.** By dividing each side of each pentagon into four equal parts, we get our template in a modified form. This time the centers of the edges of pentagons will not be used. These are just a reminder of the previous projects. We will work mostly with the new points.

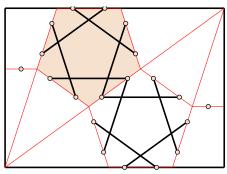
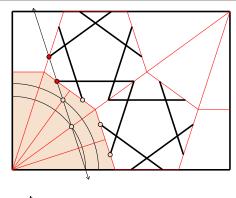
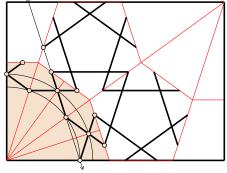


Fig. 2.58. Here I show what we can do with these new points. The midpoints of the sides of the pentagons are hidden.

If we divide the side of a pentagon into three equal parts, then the rhombus on the shared edge of two pentagons will get smaller, and the pentagons with black edges will get larger. The distance between points shown here can be larger or smaller. Each time we will get a slight variation of the same pattern.

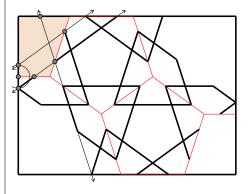


**Fig. 2.5c.** The line passing through the two red points will allow us to create a pattern inside the decagon in the bottom-left corner. After drawing the two circles passing through the intersection points of this line with symmetry lines of the decagon, we have a framework ready to draw the pattern.



**FIG. 2.5D.** Here I show how the pattern inside the decagon was created.

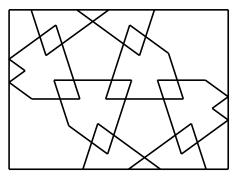
This part may have some interesting variations. One can consider this pattern as a structure of some specific overlapping kites. Interestingly, the lengths of sides of these kites have proportion L/S= 1.61803=Φ. Thus the golden ratio is still with us.

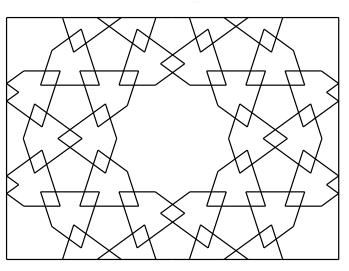


**FIG. 2.5E.F.G.** To fill the top-left corner with a pattern, we need to extend some elements of the pattern from the decagon and pentagon. We get a framework that will allow us to complete another kite in the top-left corner.

Below we show a complete template (left) as well as a pattern created with this template. Here we see the nice structure of the ring of ten kites located around the center of the design.

Bourgoin shows similar patterns for other types of symmetry – dodecagonal, and octagonal.

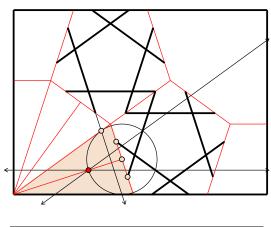




Gereh from this project is, in many aspects, different than these from previous projects. We can easily distinguish its style between many other patterns. In Tony Lee's notes, this pattern is labeled as Type VI. Unfortunately, his types IV and V are just modifications of the Type I. In the future, we will refer to it as the Tan Sahid Mosque style or simply the Tan Sahid style. This pattern may have a few interesting modifications. Here is one of them.

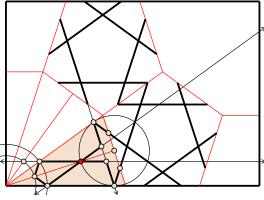
# Project 2.6 - One variation of the Tan Sahid Mosque style

In the gereh from project 2.5, we have ample empty space inside of each decagon. Such space can be a bit dull if we develop this pattern on wooden doors. For this reason, some of the woodworking masters fill it with a small rosette. This is the purpose of this project.



**FIG. 2.6A.** We draw the pattern inside pentagons precisely the same way as in project 5.

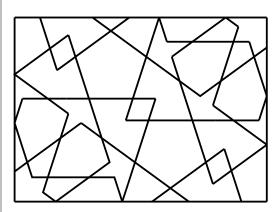
We extend one of the lines from the pattern in the pentagon, draw a circle as shown here, and then we use the red point to draw two lines parallel to the edges of the long triangle. This will be a starting point to make a small rosette inside the decagon.



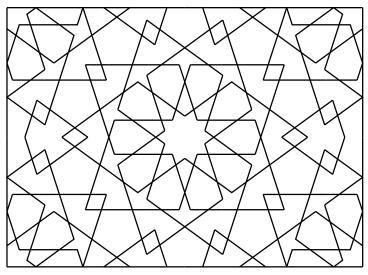
**FIG. 2.6B.** Here we have a complete pattern inside the long triangle. We have to draw an identical pattern in the remaining triangles in this decagon as well as in the one on the top-right of this image. The easiest way is to copy it to the other places using tracing paper.

Note that here we broke rule G4. Lines of the pattern from the pentagon pass the edge of the tile without changing their direction. According to rule G4, they should bend here in a mirrored direction.

In the next figure, we see a complete template and a pattern made from it.

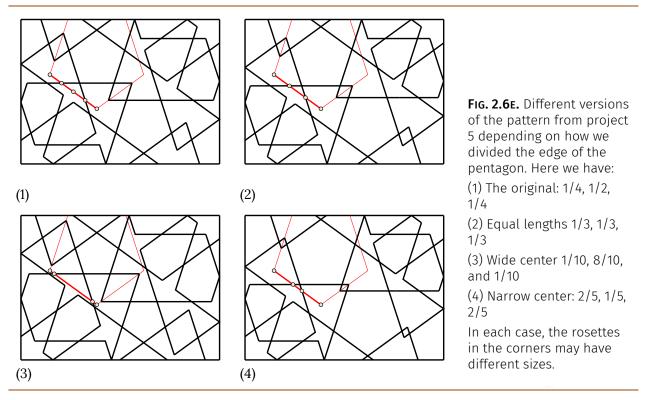


**FIG. 2.6c.D.** Template and a pattern made using four copies of this template.



A pattern identical to the one we created here can be seen in the paper by D'Avennes (2007). In the collection of Miriam and Ira D. Wallach Division of Art and Architecture in New York Public Library, we can find it with French inscription 'Art arabe: Maison de Sidi Youçouf: porte intérieure (XVIIIe. Siècle). This means Arab Art: House of Sidi Yusuf: Interior Door (Eighteenth Century).

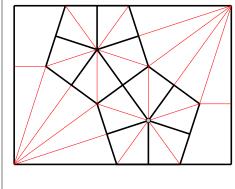
This pattern brings us to an interesting discussion. Note – while developing the pattern in pentagons, we decided to split the edge of the pentagonal tile into three parts with lengths  $\frac{1}{4}$ ,  $\frac{1}{2}$ , and  $\frac{1}{4}$  of the edge of the pentagon. This was our arbitrary decision, and one could split the edge of the pentagon into three parts with different proportions. Medieval artists often used different ratios. The resulting patterns were similar but not identical. Let us see what is possible.



The situation described here was noticed by other authors, but they never explained it adequately.

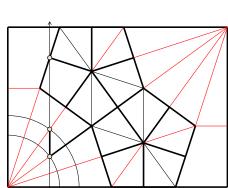
For centuries tiling and kundekari masters in Central Asia developed very specific grids that can be used for covering windows. Such grids are usually called panjara (Persian), mashrabiya (Arabic), or şebeke (Turkish). In the next project, we will create such mashrabiya using the same tessellation with two pentagons.

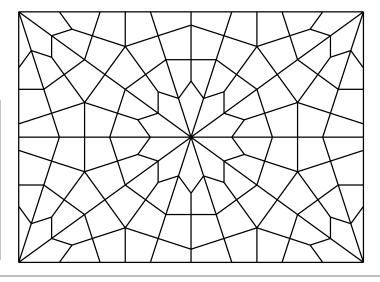
# Project 2.8 – A decagonal mashrabiya



**Fig. 2.8a.** We adjust the tessellation by adding all mirror lines for pentagons and decagons. Then we use them to draw a pattern inside the pentagons. We also can treat the tessellation edges as a part of the design.

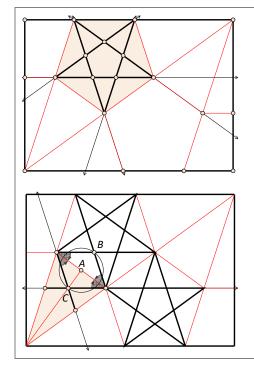
**FIG. 2.8B.C.** (below left). Here we show how the pattern was expanded inside the decagon. This part may have many different versions.





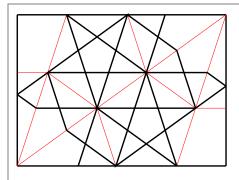
# Project 2.10 - The Sikandra style

The decagonal pattern from Akbar's mausoleum in Sikandra, in India, is somewhat unusual, and some people consider it as 'incorrect.' As we will see in a moment, this pattern precisely follows the rules of gereh.

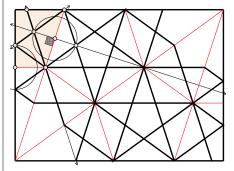


**FIG. 10A.** We start with the same tessellation as before. Then we draw the first line through the corners of the decagon. Due to the symmetry rule, we have to connect each pair of non-adjacent vertices with a segment. Then we use these lines to create a star inside the pentagon. In the original pattern from Sikandra, this star has a small pentagon in the middle, but we can also leave the star without the pentagon.

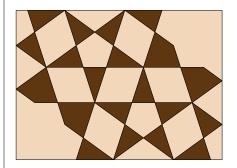
**Fig. 10B.** We fill the other pentagon with an identical star. Then using a circle with its center at the midpoint of a pentagon side, point A, and radius AB. This way, we obtain point C and the lines going through it, and the vertices of the pentagon form a rhombus. We use these lines to draw the pattern inside the long triangle. Note – we are still following the gereh rules. The angles between segments on both sides of the pentagon and decagon are equal.



**Fig. 10c.** After copying the pattern from the long triangle into other places, we should get an image identical to this one here. Now we have to add the pattern to the remaining empty spaces.

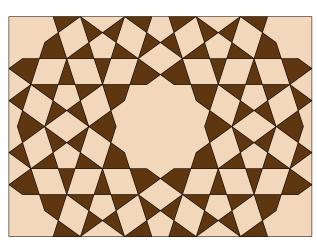


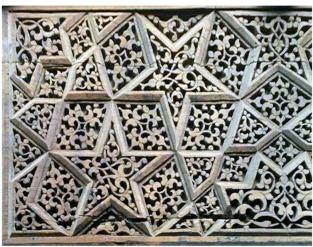
**FIG. 10D.** Here I show how the top-left empty space should be filled with the pattern. Again we use circles to get equal angles of the design on both sides of tessellation edges.



**FIG. 10E.F.G.** This image shows how the template of the pattern from Sikandra looks. Its original colors are similar to these that we have here. Note also – the large empty spaces in the left-bottom as well as in the right-top corner can be filled with additional decoration. The original pattern has such a decoration.

Below are shown two versions of the Sikandra pattern. The left one shows what we get from our template. The right figure shows the same design from Birgi Aydınoglu Mehmet Bey Mosque, Odemis, in Turkey with some extra decorations.



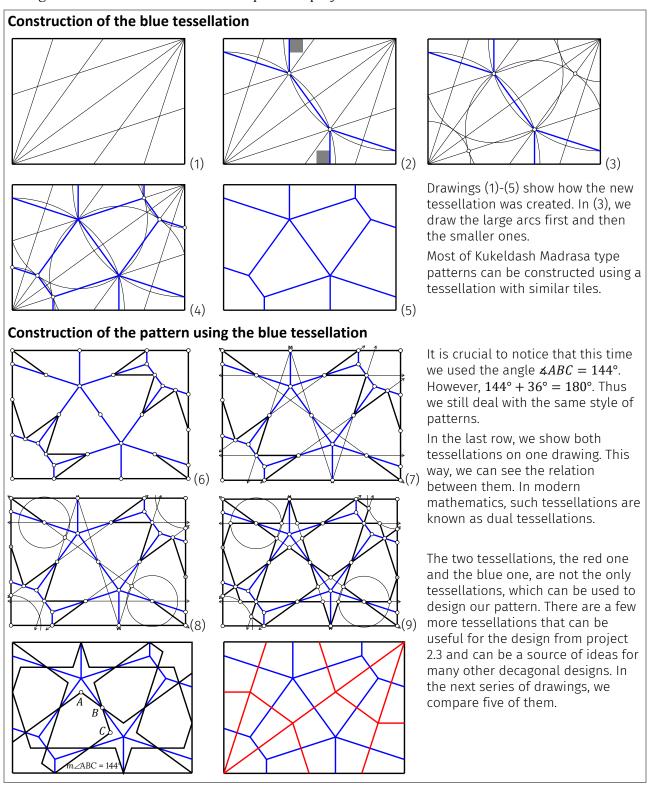


For this pattern, we used the name Sikandra style. Tony Lee calls it a Type XI pattern.

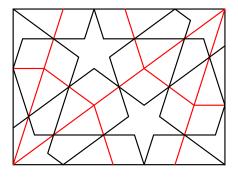
# Multiple tessellations for a decagonal pattern

Before we close this paper, let us look at different ways of creating the gereh from project 2.3. This will show us one important thing – the type 1 or Kukeldash Madrasah style can be created using different angles than in project 2.3. The only thing we have to change is the tessellation.

Construction of the new tessellation is more complicated than tessellation from each of the previous projects. For this reason, we will show a very detailed development of it. We will use the blue color to distinguish it from the tessellation from previous projects where we used the red color.



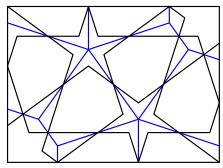
# Five tessellations for design 2.3



#### The red tessellation

As we remember, we created the pattern on this tessellation by drawing lines connecting opposite midpoints on the edges of pentagons. Designs in the remaining figures (decagons and trapeziums) are a natural consequence of this choice.

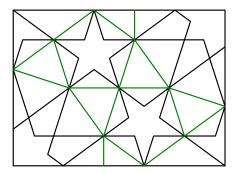
This is the most popular and most useful tessellation. We have here three polygons – a regular pentagon, a golden trapezium, and a regular decagon. The regular decagon can also be treated as a union of 10 golden triangles.



#### The blue tessellation

We created this tessellation on the previous page. For the pattern from project 2.3, its construction was straightforward. However, for a more complex pattern construction of analogical tessellation can be a more difficult task.

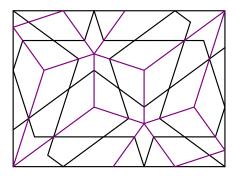
The pattern was created by connecting centers of edges of trapeziums. Note, the angles between edges of tessellation and lines of the pattern are not the same as in Devon Madrasah style.



# The green tessellation

This tessellation was derived from one of the Iranian papers on geometric pattern design. It uses regular pentagons, regular decagons, and two types of triangles. The larger triangle we can fill with the pattern, but the smaller one we leave empty (no one ever said that all tessellation tiles should be supplied with design).

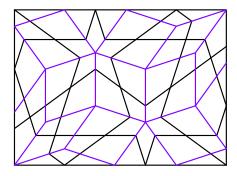
The construction of this tessellation is simple. We use the vertices of polygons to create the pattern. This is the same approach as in the Sikandra style.



# The purple tessellation

This tessellation was derived from a drawing in an old paper (1947) by Russian scientist Balkanov. Creating this tessellation is not a difficult task. The pattern was designed by drawing lines through the midpoints of long edges of kites parallel to the opposite long edges of kits.

NOTE: The flat triangles in top-left and bottom-right corners are difficult to fill with the pattern following all gereh rules. Thus we had to break one of the rules of gereh design.



#### The violet tessellation

This tessellation was invented a few years ago by a French mathematician, J.M. Castera.

In this tessellation, we have two rhombi. One is with angles  $72^{\circ}$  and  $108^{\circ}$  (fat rhombus), and another one with  $36^{\circ}$  and  $144^{\circ}$  (thin rhombus).

# **Summary**

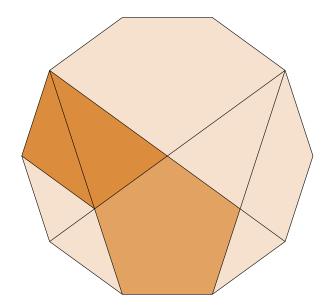
Let us summarize the information that we discussed in this paper.

**TESSELLATIONS AND PATTERNS:** The most essential point is the fact that from one tessellation, we can construct many different designs representing different types of styles. Thus to create a pattern using the gereh technique, or method, we should prepare the right contour, a convenient tessellation covering this contour, and then add a pattern to this tessellation.

In many cases, one pattern can be created using a few different tessellations. In the red and blue tessellations, we used two different angles, 144 and 36 degrees, but these two angles are connected with the equation:  $144^{\circ} + 36^{\circ} = 180^{\circ}$ . In the same way, we can create the design from project 2.4. This time we will have angles 72 and 108 degrees and, of course,  $108^{\circ} + 72^{\circ} = 180^{\circ}$ .

The red and blue tessellations in modern geometry are called dual tessellations. Tessellations green and blue are dual also. Each of them has its specific features, and as we have noticed, constructing the blue tessellation was a more difficult task than the red one. In general, we will need to create only one of these tessellations.

**GEOMETRY:** It is worth looking at the geometry of decagonal tessellations. The next drawing shows polygons that occurred in our tessellation. This is a very basic set, and it is convenient for many decagonal patterns. Later we will see other polygons that occur in decagonal tessellations.



The relation between the three polygons used in the red tessellation for this paper: decagon, pentagon, and trapezium

One can easily calculate that these three figures have edges with two lengths only. If L is half the diagonal of the decagon and S is its side, then the pentagon has sides equal to S, and the trapezium has three sides equal to S and one equal to L. The long triangle that is 1/10 of the decagon area has two sides L and one S.

**STYLES AND TYPES:** In terms of different styles, we are still able to add a lot. In terms of types – many types in Tony Lee's manuscript are simply the same types as those mentioned in this paper but using different tessellations. We will see this in a few later projects.

This discussion shows how difficult it can be any attempt at classifying geometric patterns. There is always more – we can find patterns mixing two or more different styles in one design. There are many such patterns in mosques from the Ottoman Empire. We already had such a pattern in this paper. In project 2.7, we mixed the Tan Sahid style with the Persian style.

There is also another possibility. Depending on the way how we look at a design, we may classify it in one or another style. Finally, patterns may have many modifications or special decorations that make it hard to assign them to one or another style or type.

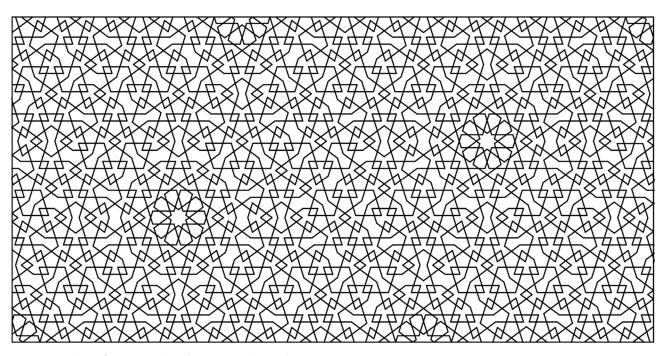
**FINAL COMMENTS:** For this paper, there are no supplementary materials. Each of the presented here projects should be created by interested readers, from scratch, using paper, pencil, compasses, and ruler. One can also use any software for high school geometry. This way, readers can better understand presented

here concepts and start experimenting on their own with geometric patterns. All this is along with the famous saying 'no pain, no gain.'

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# An example of a complex decagonal gereh

This example was created using the Tan Sahid Mosque approach. However, the pattern contains also features typical for some other styles.